Strange nonchaotic repellers

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We show the existence of strange nonchaotic repellers—that is, systems with transient dynamics whose nonattracting invariant set is fractal, but whose maximum Lyapunov coefficient is zero. We introduce the concept using a simple one-dimensional map and argue that strange nonchaotic repellers are a general phenomenon, occurring in bifurcation points of transient chaotic systems. All strange nonchaotic systems studied to date have been attractors; here, it is revealed that strange nonchaotic sets are also present in transient systems.

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Chaos in dynamical systems can be either persistent or transient. In the former case, there is an attracting set of orbits in phase space, towards which trajectories converge asymptotically. These attractors have usually convoluted, fractal geometrical structures, and they are called *strange attractors* [1]. In this context, "strange" means having a fractal geometry. In the case of transient chaos, there is also a strange (that is, fractal) invariant set in phase space, but it is nonattracting, and typical initial conditions do not converge to it. It is usually referred to as a *repeller*, or *chaotic saddle*. In both cases, chaos is usually characterized by the Lyapunov exponent, defined over the natural measure of the respective invariant sets. It was found some time ago [2,3] that there are dynamical systems with attractors which have a fractal structure, but whose Lyapunov coefficients are nonpositive. In other words, they have strange attractors, but are nonchaotic. Since then, many such strange nonchaotic attracting systems have been found [4–10] and their properties have been thoroughly investigated [11–15]. They are also physically realizable and have been observed in a number of systems [16,17]. However, to our knowledge all strange nonchaotic systems studied to date are attractors. This invites the question: are there systems with strange nonchaotic *repellers* (SNCRs)? If so, when are they expected to occur in physical systems?

We address these questions in this paper. We find that indeed there are strange nonchaotic repellers and that they appear when systems presenting transient chaos go through bifurcations. We show that in SNCRs, even though the Lyapunov exponent is zero, the escape time is a fractal function of the initial conditions, with a Cantor set of singularities where it diverges. This implies that there is a fractal boundary separating different outcomes of the transient dynamics, with a noninteger value of the fractal dimension. We argue at the end of the paper that this implies a kind of sensitivity to initial conditions in SNCRs, despite the fact that they have nonpositive Lyapunov exponents. This means that for these systems, the connection between the Lyapunov exponents and sensitivity of the dynamics to initial conditions has to be reexamined.

We present the basic ideas using as an example a simple family of one-dimensional maps.

One of the simplest and most useful dynamical systems presenting transient chaos is the well-known tent map. We want to build a map similar to the tent map, but which has a nonhyperbolic fixed point at the origin.[1] We argue below that the presence of this nonhyperbolic point causes the Lyapunov exponent to be zero. At the same time, we want to preserve the overall topology of the map, so that a complex symbolic dynamics is possible, analogously to what happens with the tent map. A natural choice of a map satisfying these conditions is given by $x_{n+1}=f(x_n)$, where

$$f(x) = x + Ax^{\alpha}, \quad \text{for } x < 1/2,$$

$$f(x) = (1-x) + A(1-x)^{\alpha}, \quad \text{for } x \geq 1/2, \tag{1}$$

with A and α being real parameters. For $\alpha=1$ this map reduces to the tent map, but for $\alpha>1$, it has the shape shown in Fig. 1. If the condition

$$A > 2^{\alpha-1} \tag{2}$$

is satisfied, there is an interval surrounding $x=1/2$ whose orbits escape the next iteration, just as for the tent map. The pre-image of this interval is clearly composed of two intervals, as shown in Fig. 1; these points escape in two iterations. Analogously, the pre-image of this set consists of four segments, which escape in three iterations, and so on. One can clearly see that if this process is continued indefinitely, the set of remaining, nonescaping points has a Cantor-like structure, albeit not a uniform one such as the one resulting from the tent map. This is due to the fact that the slope of map (1) is not constant (in modulus), as in the tent map. Nevertheless, the Cantor-set topology is clearly there, and consequently we expect the escape to have a fractal structure. In particular, we expect the escape time (the number of iterations before escaping) to have a fractal dependence on the initial conditions, with a Cantor set of points where it diverges, reflecting the Cantor set of the nonescaping points discussed above.

This is shown to be in fact the case in Fig. 2, which displays the escape time as a function of the initial value of x, calculated by numerically iterating Eq. (1) until the escape. The intricate fractal structure is clearly seen in the successive magnifications. To confirm this fractality, we cal-

[1]Throughout this paper, we refer to fixed points and periodic points as *hyperbolic* if the module of their corresponding eigenvalue is not 1 and *nonhyperbolic* otherwise.

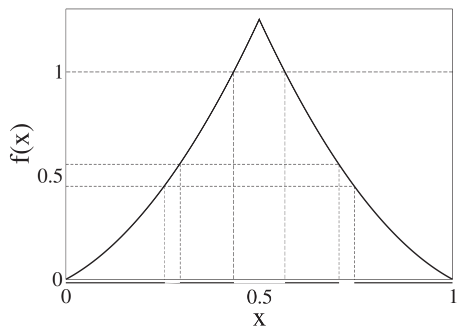

FIG. 1. Map $x_{n+1}=f(x_n)$, with $f(x)$ given by Eq. (1), for $\alpha=2$ and $A=3$. The first two steps in the formation of the fractal nonescaping set are shown.

culate the (box-counting) fractal dimension d by using the uncertainty method [18]. This method consists of choosing a large number of pairs of initial conditions, each pair separated by a distance ε and each pair lying on a random position of the segment $[0,1]$. If the escape times of the two corresponding trajectories differ, they are considered to be an *uncertain* pair. In this way, we numerically calculate the fraction $f(\varepsilon)$ of uncertain pairs as a function of the separation ε. $f(\varepsilon)$ in most cases satisfies a power law for small enough ε, of the form $f(\varepsilon)\propto\varepsilon^{1-d}$, where d is the (box-counting) fractal dimension. We can therefore determine the fractal dimension numerically (for more details, see Ref. [18]). To achieve maximum precision, we use an arbitrary-precision numerical library [19], which allows us to do the computations with higher accuracy than the usual machine precision. We were thus able to calculate $f(\varepsilon)$ for values of ε as small as 10^{-39}. The resulting scaling is shown in Fig. 3, where we see that there is indeed a well-defined power law. The fractal dimension is determined from the slope of the fitted line, and the

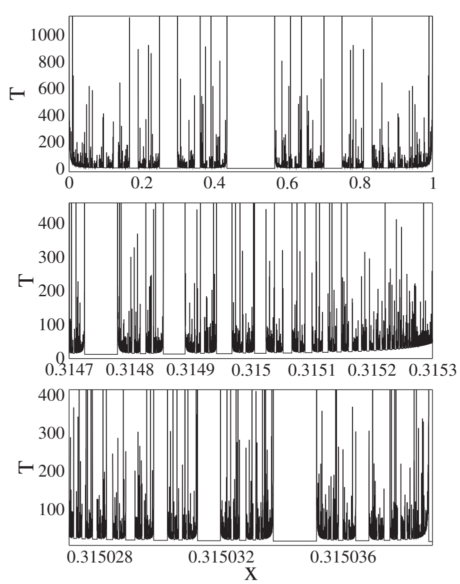

FIG. 2. Escape time T as a function of the initial position x, for the map of Eq. (1), with $\alpha=2$ and $A=3$. Successive magnifications show the fractal nature of $T(x)$.

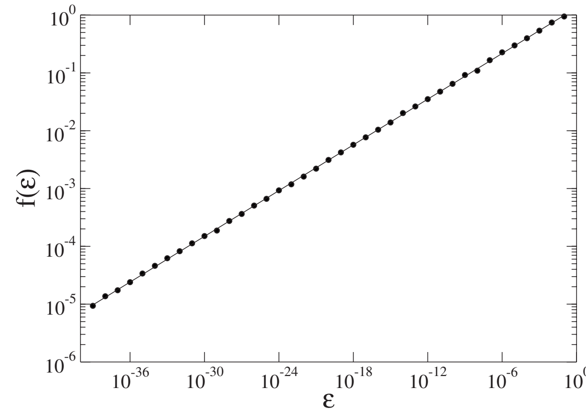

FIG. 3. Fraction $f(\varepsilon)$ of uncertain pairs as a function of their separation ε, for the map 1. The fitting to a power law is also shown, giving $f(\varepsilon)\propto\varepsilon^{0.13}$, which implies a fractal dimension of $d=0.87$.

result is $d=0.87$. We notice that, even though we only present the results for $\alpha=2$, simulations with other values of α (greater than 1) satisfying Eq. (2) always show a fractal escape.

We note that there is a relation between the behavior of maps like $f(x)$ of Eq. (1) and the phenomenon of intermittence [20]. In an intermittent transition to chaos through a saddle-node bifurcation, at the transition point the dynamics also has a nonhyperbolic fixed point and, consequently, a null Lyapunov exponent. Another similarity is that the natural measure in those intermittent systems is concentrated in one point. Beyond the transition, at the intermittent regime proper, the periods of regular behavior followed by bursts of chaotic motion can be loosely regarded as a kind of escape phenomenon, somewhat similar to what we study in this work. However, systems with intermittent chaos are not transient: it is persistent dynamics governed by a chaotic attractor, albeit with alternating periods of regular and irregular behavior. As a result, one cannot define an escape time for intermittent systems, and thus notions such as fractal escape functions and the corresponding fractal dimension (Fig. 2) are not applicable. Another system worth mentioning is the *critical Harper map* [21], which is a two-dimensional map with all Lyapunov coefficients equal to zero. Some rigorous results have been obtained for this system. It is shown in [21] that its dynamics is topologically transitive, which means that its natural measure is spread throughout the phase space, in contrast with our system.

We have seen above that there is a strange (or fractal) invariant set in the dynamics of the maps (1). We claim that the Lyapunov exponent for these maps is zero if $\alpha>1$. This is a consequence of the behavior of the map near the origin. From Eq. (1), we see that $f'(0)=1$ if $\alpha>1$. The origin is thus a nonhyperbolic fixed point, with unit eigenvalue. The Lyapunov exponent h is the average of $\ln|f'(x)|$ over the natural measure of the map: $h=\int\ln|f'(x)|d\mu(x)$. Because this is an escaping dynamics [we are always assuming condition (2) does hold], the measure μ is "sparse:" it has support on a fractal set of zero Lebesgue measure. Let I be an interval contained in $[0,1]$, and let $\mu(I)$ denote the natural measure of I. A finite-time approximation of μ is obtained by considering the position of orbits with initial conditions picked randomly in $[0,1]$, which have not escaped after many iterations. More precisely, considering the set of orbits which start from a large number of random initial conditions in $[0,1]$, let N_n be the total number of these orbits which have not escaped after $2n$ iterations. The corresponding approximate natural measure of I is given by the ratio $\mu_n(I)$, defined by $\mu_n(I)=N_n(I)/N_n$, where $N_n(I)$ is the number of orbits located in I at the nth iteration, which have not escaped after $2n$ iterations. The exact measure $\mu(I)$ is found by taking the limit $n\to\infty$.

There is an infinite number of periodic orbits in the nonescaping set of the map (1). Each of these orbits contributes to μ with some weight, which depends on the dynamics in the neighborhood of each periodic orbit. The above considerations imply that the weight of a periodic orbit in determining the Lyapunov exponent h is inversely proportional to the *total* rate of escape in their vicinity. Points near hyperbolic orbits have an exponential escape rate, determined by their eigenvalue. But points are also mapped to the vicinity of periodic orbits, coming from neighborhoods of other pre-images of the periodic points—a result of the noninvertibility of one-dimensional chaotic maps. Despite this "nonlocal" effect, we still expect the overall escape rate in the vicinity of hyperbolic orbits to be exponential,[2] as long as none of the pre-images is a nonhyperbolic point. Points in the vicinity of nonhyperbolic periodic orbits behave very differently: their escape rate is much slower, following a power law of time. The same considerations discussed above apply, and we again expect that the resulting escape rate will also be a power law.

Thus, as the number of iterations increases, the weight of a nonhyperbolic fixed point becomes ever greater and the measure μ, determined by the infinite-time limit, is dominated by these points. This suggests that the presence of a nonhyperbolic fixed point causes the Lyapunov exponent of a map like Eq. (1) to be zero. This is seen to be indeed the case by calculating finite-time approximations h_n for the Lyapunov exponent, based on the finite-time measures μ_n discussed above. This can be done numerically by using the *sprinkler method*, presented in detail in Refs. [22,23]. The result is displayed in Fig. 4(a). We see that h_n decays towards zero, approximately following $h_n \sim t^{-1}$ (from nonlinear fitting to the data shown in the figure). This shows conclusively that the Lyapunov exponent of this system is zero. The map (1) therefore has a strange nonchaotic repeller. As we saw, this behavior is a consequence of the presence of a nonhyperbolic fixed point. We have verified that the same happens for other values of α.

From the above reasoning, we should also expect that the total number of particles, N_n, which remain in the interval $[0,1]$ should decay according to a power law, in contrast to the hyperbolic case, where the decay is exponential. In fact, since the measure is so concentrated at the origin, this escape rate must be the rate at which orbits separate from the fixed point $x=0$. This is readily calculated from Eq. (1), and we find $N_n \sim t^{1/(1-\alpha)}$. The result of simulating this escape is shown in Fig. 4(b). The decay is indeed polynomial, and the power-law coefficient agrees very well with this prediction.

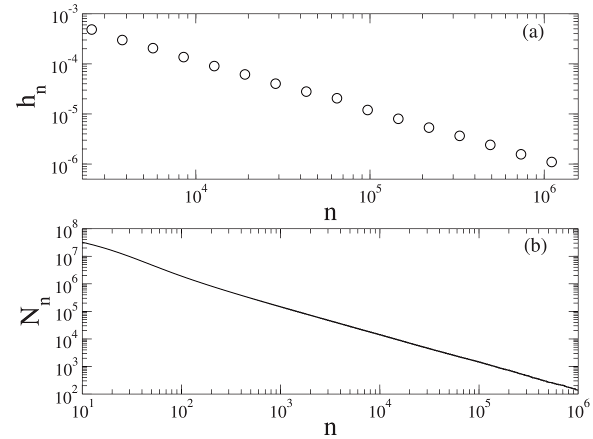

FIG. 4. (a) Finite-time approximation h_n for the Lyapunov exponent as a function of the number n of iterations. (b) Number N_n of orbits remaining in the interval $[0,1]$ after n iterations; the decay fits the law $N_n \sim t^{-1}$. The parameters of the map are in both cases $\alpha=2$, $A=3$.

Nonhyperbolic transient dynamics in one-dimensional maps has been touched upon in Refs. [24,25], where it is called a "border state of chaos." The source of nonhyperbolicity in that case is the divergence of the map's derivative at the escaping window's border for a critical parameter value. In that case, nonhyperbolicity does not lead to a zero Lyapunov exponent, as it is not caused by a fixed point with a unit eigenvalue. As a result, the escape rate is still exponential, and from the point of view of the escape dynamics it behaves as a hyperbolic chaotic scattering system, even though its measure is also singular. In other words, it is not a strange nonchaotic repeller.

As we have seen above, the strange nonchaotic saddle appears as the result of the presence of a nonhyperbolic fixed point. This suggests that SNCRs are very general, since nonhyperbolic orbits appear whenever dynamical systems undergo bifurcations.

The Lyapunov exponent has traditionally been interpreted as a measure of the sensitivity of a system to initial conditions. But in transient chaos, the sensitivity of the asymptotic behavior of the dynamics to initial conditions is more naturally represented by the fractal dimension of the repeller. In SNCRs, the fractal dimension is noninteger, even though the Lyapunov exponent is zero. The system has a dynamics which is sensitive to small perturbations of the initial conditions, because the noninteger fractal dimension of the repeller means that it is very hard to know where a trajectory will eventually go, despite the fact that the Lyapunov exponent is zero. A similar result was shown recently for strange nonchaotic attractors [26]: they also have a kind of sensitivity to initial conditions, even though they do not have positive Lyapunov exponents. This means that the Lyapunov exponent is not a good measure of sensitivity for these systems.

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